

# 1+1 Dimensional Compactifications of String Theory

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## Abstract

We argue that stable, maximally symmetric compactifications of string theory to 1+1 dimensions are in conflict with holography. In particular, the finite horizon entropies of the Rindler wedge in 1+1 dimensional Minkowski and anti de Sitter space, and of the de Sitter horizon in any dimension, are inconsistent with the symmetries of these spaces. The argument parallels one made recently by the same authors, in which we demonstrated the incompatibility of the finiteness of the entropy and the symmetries of de Sitter space in any dimension. If the horizon entropy is either infinite or zero the conflict is resolved.

# 1 Introduction

The holographic principle [1, 2] has become one of the most important ideas in quantum gravity. It has led to many major advances in the field in recent years. One may wonder whether it constrains the possible backgrounds in string theory, or any consistent theory that includes gravity. Indeed, in a recent paper [3] we have argued that the symmetries of de Sitter space are incompatible with the holographic requirement of finite entropy of a causal patch. We further argued that the incompatibility shows up in the very long time behavior of inflating spaces, perhaps as an inevitable instability.

In this note we will use the same argument to constrain the set of possible maximally symmetric compactifications of string theory to two dimensions<sup>1</sup>, at least in cases where they admit a geometric description. We will show that consistency with holography requires at least three space-time dimensions.

In [3], the authors noted the fact that in all representations of the de Sitter group, the boost generator (which generates time translations for the static patch) has a continuous spectrum. However, the area of the de Sitter horizon is finite, which implies a finite entropy for the quantum de Sitter Hamiltonian, and finite entropy is inconsistent with a continuous spectrum. Therefore, pure, eternal de Sitter space is inconsistent with holography. The simplest explanation is that all de Sitter solutions in string theory are at best metastable, with a finite lifetime followed by decay to a supersymmetric vacuum.

We can apply the same logic to 1+1 dimensional Lorentz and AdS invariant compactifications of string theory. The area of the horizon of the Rindler wedge of 1+1 dimensional Minkowski space is simply the volume of the compact internal space, and hence is finite. However, the 1+1 dimensional Poincaré algebra has the same commutation relation used in [3] to prove continuity of the spectrum. Therefore, by the same argument, the Rindler Hamiltonian is continuous, which is inconsistent with the finite area of the horizon. An identical argument can be made for 1+1 dimensional anti de Sitter space. Our conclusion is that maximally symmetric compactifications of string theory to 1+1 dimensions in which the horizon area is macroscopic are never stable. One possible resolution is that the volume of the compact space is infinite; another is that it is very small and the entropy is zero. In the latter case there is only order one state in the Hilbert space, and the symmetry argument does not apply.

It is also the case that supersymmetric compactifications to 1+1 dimensions will typ-

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<sup>1</sup>We thank Juan Maldacena for suggesting the application of the argument to AdS<sub>2</sub>.

ically contain massless moduli, which in two dimensions have an IR quantum instability. The expectation values of such moduli will quickly spread to infinity (*e.g.* if an IR cutoff is suddenly removed), which may account for the instability in the size of the compact manifold. Similar arguments were made in [4].

The plan of the paper is as follows: in section 2, we review the formalism of thermofield dynamics and its connection to the Rindler wedge of Minkowski space. In section 3, we prove (using the technique of [3]) that the Rindler Hamiltonian is continuous. The same proof applies to the Hamiltonian of the “Rindler” wedge of AdS. In section 4, we discuss the field theoretic instability, non-Lorentz invariant compactifications to 1+1, and conclude. A reader already familiar with the techniques of [3] may wish to skip to section 4.

Note added: after this paper was first posted on the arxiv, a preprint [5] appeared which presented a supersymmetric compactification of heterotic string theory to 1+1 dimensions. We believe that despite the lack of massless propagating degrees of freedom, due to IR divergences this model has a completely trivial S-matrix, and as such evades our argument in a way that will be mentioned briefly at the end of section 3. This physics will be discussed further in [6].

## 2 Thermofield dynamics

Thermofield theory was invented [7] in the context of many body theory for the purpose of simplifying the calculation of real time correlation functions in finite temperature systems. Its connection with black holes was realized by Israel [8], and elaborated in the holographic context by Maldacena [9]. In the thermofield formalism, one takes the tensor product of two copies of the original field theory, labeled by 1, 2. The two copies are decoupled, and the total Hamiltonian is

$$H_{tf} \equiv H \otimes I - I \otimes H, \quad (2.1)$$

where  $H$  is the Hamiltonian for the original theory and  $I$  is the identity operator. We then construct the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\frac{1}{2}\beta E_i} |E_i, E_i\rangle, \quad (2.2)$$

where  $|E_i, E_j\rangle = |E_i\rangle \otimes |E_j\rangle$ , and  $|E_i\rangle$  are energy eigenstates. Strictly speaking the correlation is not between identical states but between a state and its CPT conjugate. The state  $|\psi\rangle$  is a particular eigenvector of  $H_{tf}$  with eigenvalue zero. Furthermore ambiguities in the construction due to degeneracies can be resolved so that  $|\psi\rangle$  is annihilated by the

conserved generators of symmetry transformations such as angular momentum or electric charge. Correlations between subsystems 1 and 2 are due to the entanglement in  $|\psi\rangle$ .

Operators which belong to subsystem 1 have the form  $A \otimes I$ , and will be denoted  $A_1$ . Operators associated with subsystem 2 are defined in a similar manner, except with an additional rule of hermitian conjugation:

$$A_2 \equiv I \otimes A^\dagger. \quad (2.3)$$

Standard thermal correlation functions may be written as expectation values:

$$\langle \psi | A_1(0) B_1(t) | \psi \rangle. \quad (2.4)$$

As can be easily seen from the form of  $|\psi\rangle$ , 2.4 is simply the thermal expectation value of  $A(0)B(t)$ , evaluated in a thermal density matrix at inverse temperature  $\beta$ . The state counting entropy observed in subsystem 1 is the entropy of entanglement of the state  $|\psi\rangle$ . In conventional applications, no physical significance is usually attached to correlators involving both subsystems, but we can certainly define them; for example

$$\langle \psi | A_1(0) B_2(t) | \psi \rangle. \quad (2.5)$$

In the finite temperature AdS/CFT correspondence, 2.5 has a simple interpretation [9]: it corresponds to a correlator between operators on the two disconnected boundaries of the spacetime. It is not hard to see that one can compute 2.5 by analytically continuing 2.4:

$$\langle \psi | A_1(0) B_2(t) | \psi \rangle = \langle \psi | A_1(0) B_1(-t - i\beta/2) | \psi \rangle. \quad (2.6)$$

## 2.1 Rindler space

We will now consider the relationship between thermofield dynamics and quantum field theory in spaces with horizons. The simplest example is Rindler space. One plus one dimensional Minkowski space can be divided into four quadrants: I, II, III and IV (see Figure 1). Quadrants I and III consist of points separated from the origin by a space-like separation, while points in II and IV are displaced by timelike intervals. Quadrant I is Rindler space, and can be described by the metric

$$ds^2 = r^2 dt^2 - dr^2, \quad (2.7)$$

where  $r$  is proper distance from the origin, and  $t$  is the dimensionless Rindler time. The Rindler quadrant may be described by the Unruh thermal state with temperature

$$T_{rind} = \frac{1}{2\pi} = \frac{1}{\beta_{rind}}. \quad (2.8)$$

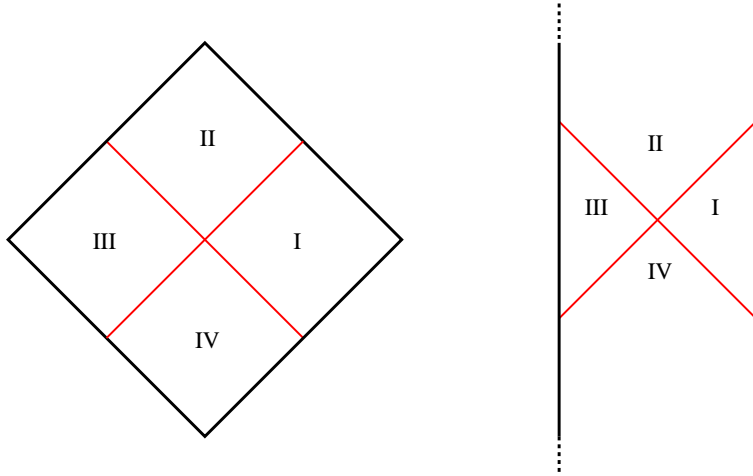


Figure 1: On the left, the conformal diagram for Minkowski space, showing the Rindler wedges I and III. In Rindler coordinates, continuing from region I to region III involves shifting time by  $i\beta/2$ , in accord with 2.6. On the right, the “Rindler” wedge of  $\text{AdS}_2$ .

Quadrant III is a copy of quadrant I, and can be precisely identified with the other half of the thermofield double. To see this we first of all note that the Rindler Hamiltonian is the boost generator. Since the Minkowski vacuum is boost invariant, it is an eigenvector of the boost generator with vanishing eigenvalue. Furthermore, the Minkowski vacuum is an entangled state of the degrees of freedom in the two quadrants I and III. Finally, it is well known that when the density matrix for quadrant I is obtained by tracing over III the result is a thermal state at the Rindler temperature. It is easy to see that correlators between fields in quadrants I and III are related by exactly the same analytic continuations derived from thermofield dynamics. To see this, recall that the usual Minkowski variables  $X^0, X^1$  are related to the Rindler coordinates by

$$\begin{aligned} X^0 &= r \sinh t \\ X^1 &= r \cosh t. \end{aligned} \tag{2.9}$$

Since the inverse Rindler temperature 2.8 is  $2\pi$ , the continuation in equation 2.6 is

$$t \rightarrow t - i\pi, \tag{2.10}$$

or, from 2.9,  $X^\mu \rightarrow -X^\mu$ . Thus the thermofield continuation takes quadrant I to quadrant III.

In any dimension other than 1+1 the area of the horizon—and therefore the entropy of Rindler space—is infinite. However, if we consider the space  $\mathcal{R}_{1,1} \times \mathcal{M}_8$ , where  $\mathcal{M}_8$  is

any eight-dimensional compact manifold, the area of the Rindler horizon is nothing but  $\text{vol}(\mathcal{M}_8)$ , and therefore the entropy of the Rindler wedge must be finite.

A very similar construction produces the “Rindler” wedge of anti de Sitter. Start with the coordinates in which Euclidean AdS is a ball:  $ds^2 = d\rho^2 + \sinh^2 \rho d\Omega^2$ . Continuing the azimuthal angle of the sphere  $\phi \rightarrow it$  gives a metric covering one quarter of the AdS hyperboloid (see figure 1). As in flat space, the horizon area is finite only for the case of  $\text{AdS}_2$ .

### 3 Continuity of the spectrum

It is a fact that any system with finite entropy must have a discrete spectrum; actually, the statement is stronger: there must be a finite number of states below any given energy  $E$ , so the spectrum can not have discrete accumulation points. Because this is central to our analysis, we present a short proof below.

The entropy  $S$  is defined by  $S \equiv -\text{Tr} \rho \ln \rho$ , where  $\rho \equiv e^{-\beta H}/Z$  and  $Z \equiv \text{Tr} e^{-\beta H}$ . Here trace means the sum over all the states in the spectrum of the Hamiltonian, to be replaced by an integral with the proper measure in the case that the spectrum is continuous. Expanding the relation above gives

$$S = (1/Z) \text{Tr} e^{-\beta H} (\beta H + \ln Z) = \ln Z + (\beta/Z) \text{Tr} H e^{-\beta H}, \quad (3.11)$$

which is nothing but the ordinary thermodynamic relation  $S = -\beta F + \beta E$ . As can be easily seen from 3.11,  $S$  is not affected by the shift  $H \rightarrow H - E_0$ , for any constant  $E_0$ . Therefore we can take the ground state energy to be zero (we will assume the energy is bounded from below). Then the second term in 3.11 is manifestly positive, and

$$S \geq \ln Z = \ln \sum e^{-\beta H} \geq \ln (N e^{-\beta E_N}), \quad (3.12)$$

where  $E_N$  is the  $N$ th energy level, and we have temporarily assumed that the spectrum is countable. Now the proof is clear: if there exists any energy  $E_N$  below which there are an infinite number of states, the entropy will diverge; and hence finite entropy implies a discrete spectrum with no accumulation points.<sup>2</sup>

This demonstrates that if the entropy of the Rindler wedge is finite, then the Rindler Hamiltonian  $H_R$  must have a discrete spectrum. The implications for the thermofield

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<sup>2</sup>Note that this does *not* imply that the Hilbert space is finite. In general, the level spacing  $\delta E \sim T e^{-S}$ , at least for  $T \gg \delta E$ .

Hamiltonian  $H_{tf}$  are weaker. The spectrum of this operator consists of the differences of eigenvalues of  $H_R$ , namely  $E_i - E_j$ . If the spectrum of  $H_R$  is discrete, this set of numbers need not be, but is certainly countable. In the next section we will prove that the spectrum of the full thermofield Hamiltonian  $H_{tf}$  is not countable, and hence that the entropy of the thermal ensemble defined using  $H_R$  is infinite.

### 3.1 Representations of the Poincaré algebra

The Poincaré group in 1+1 dimensions has three generators, which satisfy the following algebra:

$$\begin{aligned} [P_0, K] &= iP_1 \\ [P_1, K] &= iP_0 \\ [P_0, P_1] &= 0. \end{aligned} \tag{3.13}$$

The  $P_i$  generate translations in space and time, and  $K$  is the boost generator. The choice of a Rindler wedge preserves only the generator  $K$  (this is obvious if one recalls that the Rindler wedge is the region of space seen by a uniformly accelerating observer). The boost generator  $K$  acts by generating translations in Rindler time  $t$ ; in other words, the Rindler Hamiltonian is  $K$ . The generators  $P_i$  do not act on the Hilbert space of one wedge; rather, they act on the full thermofield double space by mixing degrees of freedom from the two halves. In the case of a free field theory this means that the modes of one Rindler wedge become a linear combination of modes of both wedges under the action of the  $P_i$ ; see the appendix of [3] for more details. The crucial point is that the states of the thermofield double space, namely Minkowski space, form a representation of the full group.

If we define  $P_{\pm} = P_0 \pm P_1$ , and identify the boost  $K$  with the thermofield Hamiltonian  $H$  the algebra becomes:<sup>3</sup>

$$\begin{aligned} [H, P_+] &= -iP_+ \\ [H, P_-] &= iP_- \\ [P_+, P_-] &= 0. \end{aligned} \tag{3.14}$$

Using 3.14, we wish to prove that the spectrum of  $H$  is continuous. Following [3], consider the operator  $e^{iP_-}(t)$ :

$$e^{iP_-}(t) \equiv e^{iHt} e^{iP_-} e^{-iHt} = e^{e^{iHt} iP_- e^{-iHt}} = e^{iP_-} e^{-t}. \tag{3.15}$$

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<sup>3</sup>For simplicity of notation, for the rest of the section we will refer to the full thermofield double Hamiltonian  $H_{tf}$  simply as  $H$ .

We will now assume that the spectrum of  $H$  is countable, and use the assumption to derive a contradiction. We have

$$\left| \langle \alpha | e^{iP_-} | \alpha \rangle \right| = 1 - \delta. \quad (3.16)$$

Here  $|\alpha\rangle$  is some state in the Hilbert space, and  $\delta > 0$  because  $e^{iP_-}$  is unitary and  $P_-$  is non-zero. Define

$$F(t) \equiv \langle \alpha | e^{iP_-(t)} | \alpha \rangle = \langle \alpha | e^{iHt} e^{iP_-} e^{-iHt} | \alpha \rangle = \langle \alpha | e^{iP_- e^{-t}} | \alpha \rangle. \quad (3.17)$$

From this,  $F(t) \rightarrow 1$  as  $t \rightarrow \infty$ , and  $F(0) = 1 - \delta < 1$ . We will now prove that  $F(t)$  is quasiperiodic (see e.g. the appendix of [12]).

Any discrete sum of the form

$$\sum_{n=1}^{\infty} f_n e^{i\omega_n t} \quad (3.18)$$

is quasiperiodic if

$$\sum_{n=1}^{\infty} |f_n|^2 < \infty. \quad (3.19)$$

Therefore, it suffices to show that  $F(t)$  can be written as a sum of this form. But, expanding the state  $|\alpha\rangle$  in the energy basis:<sup>4</sup>

$$F(t) = \sum_{n,m} f_n^* f_m \langle n | e^{iP_-} | m \rangle e^{i(\omega_n - \omega_m)t}. \quad (3.20)$$

Consider the sum

$$\sum_{m,n} f_n^* f_m f_m^* f_n \langle n | e^{iP_-} | m \rangle \langle m | e^{-iP_-} | n \rangle = \sum_n |f_n|^2 \sum_m |f_m|^2 \langle n | e^{iP_-} | m \rangle \langle m | e^{-iP_-} | n \rangle. \quad (3.21)$$

Considering the inner sum, we have (since  $\sum_m \langle n | e^{iP_-} | m \rangle \langle m | e^{-iP_-} | n \rangle = 1$ , and the terms are real and positive)

$$\sum_m |f_m|^2 \langle n | e^{iP_-} | m \rangle \langle m | e^{-iP_-} | n \rangle \leq 1, \quad (3.22)$$

and therefore

$$\sum_{m,n} f_n^* f_m f_m^* f_n \langle n | e^{iP_-} | m \rangle \langle m | e^{-iP_-} | n \rangle \leq 1. \quad (3.23)$$

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<sup>4</sup>The energies  $\omega_i$  are the eigenvalues of the thermofield Hamiltonian  $H = H_{tf}$ , and hence are *differences* of pairs of energies of the Rindler Hamiltonian  $H_R$ ; but see the last paragraph before Section 3.1.



This shows that  $F(t)$  satisfies the criterion 3.19, and hence  $F(t)$  is quasiperiodic. Therefore, since  $F(0) < 1$ ,  $F(t)$  can not tend to 1 as  $t \rightarrow \infty$ , and we have a contradiction.

We can make almost exactly the same argument using the “Rindler” wedge of  $\text{AdS}_2$ . The algebra is  $SO(2, 1)$  (which is identical to the  $\text{dS}_2$  algebra) and in fact the analogy to the argument of [3] is exact. The generator of “Rindler” time translations is one of the boost generators of the  $\text{AdS}$  hyperboloid. Again, the argument shows the the spectrum must be continuous.

This proves that  $H_{tf}$  can not have a countable spectrum, and therefore that the  $E_i$  cannot be discrete and the entropy cannot be finite. However, the area of the Rindler horizon in 1+1 dimensions is finite, and so we see there is a fundamental conflict between the holographic principle and the existence of a stable compactification of string theory to 1+1 dimensional Minkowski,  $\text{AdS}$ , or  $\text{dS}$  space.

We note one possible loophole—if the representation is trivial, so that there are only vacua in the spectrum, our argument fails. In that case the generators are zero, and there is no conflict with the algebra. We believe the examples presented in [5] fall into this category, as any state with non-zero energy will cause an infinite back-reaction on the geometry.

## 4 String theory arguments

Let us begin with supersymmetric compactifications of string theory. In general there will always be massless moduli such as the overall size of the compact space. These massless degrees of freedom can be thought of as 1 + 1 dimensional scalar fields. But it is well known that in 1 + 1 dimensions there are logarithmic infrared divergences. For example the two point function has a divergence of the form  $\int dk/k$ . While the UV divergence can of course be regulated, the IR divergence represents a true physical effect; namely that the fields fluctuate more and more at longer and longer wavelengths. In particular, if the Hamiltonian includes a mass term  $m^2 \theta(-t)$ , where  $\theta$  is the unit step function, the expectation value  $\langle \phi^2(t) \rangle$  will tend to infinity like  $t$ , for  $t > 0$ . This means that the field, once released from its confining potential, fluctuates more and more in field space. This effect indicates a decompactification of the manifold. In other words the compactification is unstable.

This argument (which was discussed previously in [4]) is in some ways more generic than the argument presented above, because it does not rely on maximal symmetry. On the other hand, it does require that at least some of the compactification moduli are

massless.

There are, of course,  $1 + 1$  dimensional compactifications of string theory such as the linear dilaton vacua in which the dilaton varies linearly, either with respect to time or space. However, these obviously violate Lorentz invariance. What is more, at the weak coupling end the volume of the compact space diverges if expressed in Planck units. Other possible examples, such as pp wave in anti de Sitter space, can be thought of as Lorentz non-invariant  $1 + 1$  dimensional theories and do not contradict our conclusion.

## 5 Conclusion

In both [3] and the present paper we have found that the delicate symmetries that ensure the equivalence of different observers (observer complementarity) can not be implemented for systems of finite entropy. It seems that for these symmetries to be exact, an infinite horizon area must be available for information to spread out in. We think that this is a general rule: Exact observer complementarity is only possible if the horizon is infinite in extent.

This raises the question of finite mass black hole horizons. In this case there is no exact symmetry between observers outside the black hole and those which fall through the horizon. Indeed sufficiently careful measurements of tidal forces should be able to tell a freely falling observer exactly when she crosses the horizon. Only in the limit of infinite mass does the horizon become precisely undetectable.

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